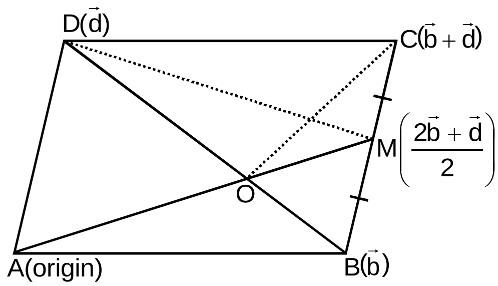
Ex.26 Given a parallelogram ABCD with area 12 sq. units. A straight line is drawn through the mid point M of the side BC and the vertex A which cuts the diagonal BD at a point 'O'. Use vectors to determine the area of the quadrilateral OMCD.

Sol. p.v. of M are $\frac{2\vec{b}+\vec{d}}{2}$



$\Delta AOD \sim \Delta OBM$

$\therefore \quad \frac{OB}{OD} = \frac{OM}{OA} = \frac{1}{2}$

p.v. of 'O' are $\frac{2\vec{b}+\vec{d}}{3}$

Given $|\vec{b}\times\vec{d}| = 12$

Area of quad. OMCD = $\frac{1}{2}|\vec{d}_1\times\vec{d}_2| = \frac{1}{2}\left|\left(\vec{d}-\frac{2\vec{b}+\vec{d}}{2}\right)\times\left(\frac{2\vec{b}+\vec{d}}{3}-(\vec{b}+\vec{d})\right)\right| = \frac{1}{12}\left|5\vec{b}\times\vec{d}\right|$ = 5 sq. units

Aliter : area ΔBCD = 6 $\Rightarrow$ $\frac{1}{2}|(\vec{c}-\vec{b})\times(\vec{d}-\vec{b})| = 6 \Rightarrow$ $|(\vec{c}-\vec{b})\times(\vec{d}-\vec{b})| = 12$

$\therefore$ p.v. of 'O' are $\frac{2\vec{b}+\vec{d}}{3}$ & 'M' are $\frac{\vec{b}+\vec{c}}{2}$

area Δ OMB = $\frac{1}{2}|\overrightarrow{BM}\times\overrightarrow{BO}| = \frac{1}{2}\left|\left(\frac{\vec{c}-\vec{b}}{2}\right)\times\left(\frac{\vec{d}-\vec{b}}{3}\right)\right| = \frac{1}{12}\times 12 = 1$

area Δ BCD – area ΔOMB = 6 – 1 = 5 sq. units

H. SHORTEST DISTANCE BETWEEN TWO LINES

If two lines in space intersect at a point, then obviously the shortest distance between them is zero. Lines which do not intersect & are also not parallel are called **skew lines**. For Skew lines the direction of the shortest distance would be perpendicular to both the lines. The magnitude of the shortest distance vector would be equal to that of the projection of $\overrightarrow{AB}$ along the direction of the line of shortest distance, $\overrightarrow{LM}$ is parallel to $\vec{p}\times\vec{q}$

i.e. $\overrightarrow{LM}$ = |Projection of $\overrightarrow{AB}$ on $\overrightarrow{LM}$| = |Projection of $\overrightarrow{AB}$ on $\vec{p}\times\vec{q}$| = $\left|\frac{\overrightarrow{AB}.(\vec{p}\times\vec{q})}{|\vec{p}\times\vec{q}|}\right| = \left|\frac{(\vec{b}-\vec{a}).(\vec{p}\times\vec{q})}{|\vec{p}\times\vec{q}|}\right|$

1. The two lines directed along $\vec{p}$ & $\vec{q}$ will intersect only if shortest distance = 0 i.e. $(\vec{b}-\vec{a}).(\vec{p}\times\vec{q}) = 0$ i.e. $(\vec{b}-\vec{a})$ lies in the plane containing $\vec{p}$ & $\vec{q}$. $\Rightarrow$ $[(\vec{b}-\vec{a})\ \vec{p}\ \vec{q}] = 0$.

2. If two lines are given by $\vec{r}_1 = \vec{a}_1 + K\vec{b}$ & $\vec{r}_2 = \vec{a}_2 + K\vec{b}$ i.e. they are parallel then, d = $\left|\frac{\vec{b}\times(\vec{a}_2-\vec{a}_1)}{|\vec{b}|}\right|$

I. PRODUCT OF THREE VECTORS

(a) SCALAR TRIPLE PRODUCT : Scalar triple product, $\vec{a} \cdot (\vec{b} \times \vec{c})$. Since the cross product $\vec{b} \times \vec{c}$ is itself a vector, we may form with it and a third vector $\vec{a}$ the scalar product $\vec{a} \cdot (\vec{b} \times \vec{c})$, which is a number. Such products of three vectors are of frequent occurrence, and we shall find it useful to examine their properties. Consider the parallelopied whose concurrent edges OA, OB, OC have the lengths and directions of the vectors $\vec{a}, \vec{b}, \vec{c}$ respectively. Then the vector $\vec{b} \times \vec{c}$, which we may denote by $\vec{n}$, is perpendicular to the face OBDC, and its modulus n is the measure of the area of that face. If θ is the angle between the directions of $\vec{n}$ and $\vec{a}$, the triple product

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = an \cos\theta = \pm V, \qquad ......(1)$$

where V is the measure of the volume of the parallelepiped. The triple product is positive if θ is acute, that is if $\vec{a}, \vec{b}, \vec{c}$ form a right-handed system of vectors.

The same reasoning shows that each of the products $\vec{b} \cdot (\vec{c} \times \vec{a})$ and $\vec{c} \cdot (\vec{a} \times \vec{b})$ has the same value ± V, being positive if the system $\vec{a}, \vec{b}, \vec{c}$ is right handed, negative if left-haded. The cyclic order $\vec{a}, \vec{b}, \vec{c}$ is maintained in each of these. If, however that order is changed, the sign of the product is changed; for $\vec{b} \times \vec{c} = -\vec{c} \times \vec{b}$. Thus

$$\pm V = \vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{b} \times \vec{c}) \cdot \vec{a} = -(\vec{c} \times \vec{b}) \cdot \vec{a} = -\vec{a} \cdot (\vec{c} \times \vec{b})$$

$$\vec{b} \cdot (\vec{c} \times \vec{a}) = (\vec{c} \times \vec{a}) \cdot \vec{b} = -(\vec{a} \times \vec{c}) \cdot \vec{b} = -\vec{b} \cdot (\vec{a} \times \vec{c})$$

$$\vec{c} \cdot (\vec{a} \times \vec{b}) = (\vec{a} \times \vec{b}) \cdot \vec{c} = -(\vec{b} \times \vec{a}) \cdot \vec{c} = -\vec{c} \cdot (\vec{b} \times \vec{a}).$$

Thus the value of the product depends on the cyclic order of the factors, but is indpendent of the position of the dot and cross. These may be interchanged at pleasure. It is usual to denote the above product by $[\vec{a}\,\vec{b}\,\vec{c}]$ or $[\vec{a}, \vec{b}, \vec{c}]$, which indicate the three factors and their cyclic order. Then

$$[\vec{a}\,\vec{b}\,\vec{c}] = -[\vec{a}\,\vec{c}\,\vec{b}]. \qquad ....(2)$$

If the three vectors are coplanar their scalar triple product is zero. For $\vec{b} \times \vec{c}$ is then perpendicular to $\vec{a}$, and their scalar product vanishes. Thus the vanishing of $[\vec{a}\,\vec{b}\,\vec{c}]$ is the condition that the vectors should be coplanar. If two of the vectors are parallel this condition is satisfied. In particular, if two of them are equal the product is zero.

There is a very simple and convenient expression for the product $[\vec{a}\,\vec{b}\,\vec{c}]$ in terms of rectangular components of the vectors.

With the usual notation we have $\vec{b} \times \vec{c} = (b_2c_3 - b_3c_2, b_3c_1 - b_1c_3 \,.\, b_1 c_2 - b_2c_1)$,

and $\vec{a} \cdot (\vec{b} \times \vec{c}) = a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ ......(3)

This is the well known expression for the volume of a parallelopiped with one corner at the origin. More generally, if in terms of three non-coplanar vectors $\vec{\ell}, \vec{m}, \vec{n}$ we write $\vec{a} = a_1\vec{\ell} + a_2\vec{m} + a_3\vec{n}$.

and so on, it is easily shown that $[\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} [\vec{\ell}\,\vec{m}\,\vec{n}]$.

The product $[\hat{i}\,\hat{j}\,\hat{k}]$, of three rectangular unit vectors, is obviously equal to unity.

Lastly, since the distributive law holds for both scalar and vector products, it holds for the scalar triple product. For instance $[\vec{a}, \vec{b} + \vec{d}, \vec{c} + \vec{e}] = [\vec{a}\,\vec{b}\,\vec{c}] + [\vec{a}\,\vec{b}\,\vec{e}] + [\vec{a}\,\vec{d}\,\vec{c}] + [\vec{a}\,\vec{d}\,\vec{e}]$,
the cyclic order of the factors being preserved in each term.

Remark :

(i) The scalar triple product of three vectors $\vec{a}.\vec{b}$ & $\vec{c}$ is defined as : $(\vec{a}\times\vec{b}).\vec{c} = |\vec{a}||\vec{b}||\vec{c}|$ $\sin\theta\cos\phi$ where θ is the angle between $\vec{a}$ & $\vec{b}$ & ϕ is the angle between $\vec{a}\times\vec{b}$ & $\vec{c}$. It is also defined as $[\vec{a}\ \vec{b}\ \vec{c}]$, spelled as box product.

(ii) Scalar triple product geometrically represents the volume of the parallelopiped whose three coterminous edges are represented by $\vec{a}, \vec{b}$ & $\vec{c}$ i.e. $V = [\vec{a}\ \vec{b}\ \vec{c}]$

(iii) In a scalar triple product the position of dot & cross can be interchanged i.e.

$\vec{a}.(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b}).\vec{c}$ OR $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$

(iv) $\vec{a}.(\vec{b}\times\vec{c}) = -\vec{a}.(\vec{c}\times\vec{b})$ i.e. $[\vec{a}\ \vec{b}\ \vec{c}] = -[\vec{a}\ \vec{c}\ \vec{b}]$

(v) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$; $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ & $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ then $[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

In general, if $\vec{a} = \vec{a} = a_1\vec{l} + a_2\vec{m} + a_3\vec{n}; \vec{b} = b_1\vec{l} + b_2\vec{m} + b_3\vec{n}$ & $\vec{c} = c_1\vec{l} + c_2\vec{m} + c_3\vec{n}$

then $[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} [\vec{l}\ \vec{m}\ \vec{n}]$; where $\vec{\ell}$; $\vec{m}$ & $\vec{n}$ are non coplanar vectors.

(vi) If $\vec{a}, \vec{b}, \vec{c}$ are coplanar $\Leftrightarrow [\vec{a}\ \vec{b}\ \vec{c}] = 0$.

(vii) Scalar product of three vectors, two of which are equal or parallel is 0 i.e. $[\vec{a}\ \vec{b}\ \vec{c}] = 0, [\vec{a}\ \vec{b}\ \vec{c}] < 0$ for left handed system.

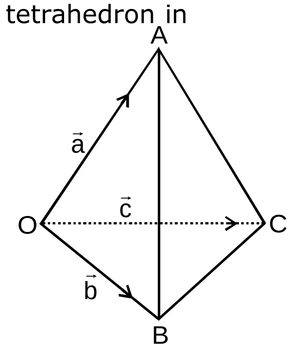
(viii) $[K\vec{a}\ \vec{b}\ \vec{c}] = K[\vec{a}\ \vec{b}\ \vec{c}]$

(ix) $[(\vec{a}+\vec{b})\ \vec{c}\ \vec{d}] = [\vec{a}\ \vec{c}\ \vec{d}] + [\vec{b}\ \vec{c}\ \vec{d}]$

(x) $[\vec{a}-\vec{b}\ \ \vec{b}-\vec{c}\ \ \vec{c}-\vec{a}] = 0$ & $[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}] = 2[\vec{a}\ \vec{b}\ \vec{c}]$.

Tetrahedron : With one vertex O as origin, let the other vertices A, B, C be the points $\vec{a}, \vec{b}, \vec{c}$ respectively. Then the vector area of OBC is $\frac{1}{2}\vec{b}\times\vec{c}$, and the value of the tetrahedron in

$$V = \left|\frac{1}{3}\vec{a}.\left(\frac{1}{2}\vec{a}\times\vec{c}\right)\right| = \frac{1}{6}|[\vec{a}\ \vec{b}\ \vec{c}]|.$$



Suppose we required the length p of the common perpendicular to the two edges AB, OC. The directions of these lines are those of the vectors $\vec{b}-\vec{a}$ and $\vec{c}$, while $\vec{a}, \vec{c}$ are two points, one on each line. If θ is their angle of inclination, $P = \dfrac{[\vec{b}-\vec{a}, \vec{c}, \vec{a}-\vec{c}]}{AB.OC.\sin\theta}$.

The numerator of this expression reduces to $[\vec{a}\ \vec{b}\ \vec{c}]$. Hence the relation $V = \frac{1}{6}AB.OC.|p|.\sin\theta$

The volume of a tetrahedron whose vertices are the points $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ is the modulus of $\frac{1}{6}[\vec{a}-\vec{d}, \vec{b}-\vec{d}, \vec{c}-\vec{d}]$. The position vector of the centroid of a tetrahedron if the pv's of its angular vertices are $\vec{a}, \vec{b}, \vec{c}$ & $\vec{d}$ are given by $\frac{1}{4}(\vec{a}+\vec{b}+\vec{c}+\vec{d})$.

Note : This is also the point of concurrency of the lines joining the vertices to the centroids of the opposite faces and is also called the centre of the tetrahedron. In case the tetrahedron is regular it is equidistant from the vertices and the four faces of the tetrahedron.

Ex.27 If $a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, b_1\hat{i} + b_2\hat{j} + b_3\hat{k}, c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ are three mutually perpendicular unit vectors, prove that $a_1\hat{i} + b_1\hat{j} + c_1\hat{k}, a_2\hat{i} + b_2\hat{j} + c_2\hat{k}, a_3\hat{i} + b_3\hat{j} + c_3\hat{k}$ are also mutually perpendicular unit vectors.

Sol. Let the three given unit vectors be $\hat{a}$, $\hat{b}$ and $\hat{c}$. Since they are mutually perpendicular $\hat{a} \cdot (\hat{b} \times \hat{c}) = 1$

$$\Rightarrow \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 1 \Rightarrow \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 1 \Rightarrow a_1\hat{i} + b_1\hat{j} + c_1\hat{k}, a_2\hat{i} + b_2\hat{j} + c_2\hat{k}, a_3\hat{i} + b_3\hat{j} + c_3\hat{k} \text{ are mutually perpendicular.}$$

Ex.28 If V be the volume of a tetrahedron & V' be the volume of the tetrahedron formed by the centroids then find the ratio of V & V'

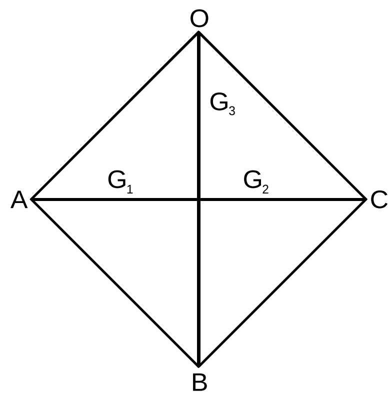
Sol. $\vec{V} = \frac{1}{6}[\vec{a}\,\vec{b}\,\vec{c}]$;

$$\vec{G} = \frac{\vec{a}+\vec{b}+\vec{c}}{3}\,;\ \vec{G}_1 = \frac{\vec{a}+\vec{b}}{3}\,;\ \vec{G}_2 = \frac{\vec{b}+\vec{c}}{3}\,;\ \vec{G}_3 = \frac{\vec{a}+\vec{c}}{3}$$

$$\overrightarrow{G_1G} = \frac{\vec{c}}{3}\,;\ \overrightarrow{G_2G} = \frac{\vec{a}}{3}\ \ \overrightarrow{G_3G} = \frac{\vec{b}}{3}$$

Hence V' = volume of tetrahedron $GG_1\,G_2\,G_3$

$$= \frac{1}{6}\left[\overrightarrow{GG_1}\ \overrightarrow{GG_2}\ \overrightarrow{GG_3}\right] = \frac{1}{6}\cdot\frac{1}{27}\,[\,\vec{a}\,\vec{b}\,\vec{c}\,] = \frac{V}{27}$$



(b) VECTOR TRIPLE PRODUCT

Consider next the cross product of $\vec{a}$ and $\vec{b} \times \vec{c}$, viz. $\vec{p} = \vec{a} \times (\vec{b} \times \vec{c})$.

This is a vector perpendicular to both $\vec{a}$ and $\vec{b} \times \vec{c}$. But $\vec{b} \times \vec{c}$ is normal to the plane of $\vec{b}$ and $\vec{c}$, so that $\vec{p}$ must lie in this plane. It is therefore expressible in terms of $\vec{b}$ and $\vec{c}$ in the form $\vec{p} = \ell\vec{b} + m\vec{c}$.

To find the actual expression for $\vec{p}$ consider unit vectors $\hat{j}$ and $\hat{k}$, the first parallel to $\vec{b}$ and the second perpendicular to it in the plane $\vec{b}$, $\vec{c}$. Then we may put $\vec{b} = b_1\hat{j}$, $\vec{c} = c_2\hat{j} + c_3\hat{k}$.

In terms of $\hat{j}$, $\hat{k}$ and the other unit vector $\hat{i}$ of the right-handed system, the remaining vector $\vec{a}$ may be written $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$. Then $\vec{b} \times \vec{c} = bc_3\hat{i}$, and the triple product

$$\vec{a} \times (\vec{b} \times \vec{c}) = a_3b_1c_3\hat{j} - a_2b_1c_3\hat{k} = (a_2c_2 + a_3c_3)\,b_1\hat{j} - a_2b_1(c_2\hat{j} + c_3\hat{k}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} \quad ...(1)$$

This is the required expression for $\vec{p}$ in terms of $\vec{b}$ and $\vec{c}$.

Similarly the triple product $(\vec{b} \times \vec{c}) \times \vec{a} = -\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}\cdot\vec{b})\vec{c} - (\vec{a}\cdot\vec{c})\vec{b}$. .....(2)

It will be noticed that the expansions (1) and (2) are both written down by the same rule. Each scalar product involves the factor outside the bracket; and the first is the scalar product of the extremes.

In a vector triple product the position of the brackets cannot be changed without altering the value of the product. For $(\vec{a} \times \vec{b}) \times \vec{c}$ is a vector expressible in terms of $\vec{a}$ and $\vec{b}$; $\vec{a} \times (\vec{b} \times \vec{c})$ is one expressible in terms of $\vec{b}$ and $\vec{c}$. The products in general therefore represent different vectors.

If a vector r is resolved into two others in the plane of $\vec{a}$ and $\vec{r}$, one parallel to $\vec{a}$ and the other perpendicular to it, the former is $\frac{\vec{a}.\vec{r}}{\vec{a}^2}\vec{a}$, and therefore the latter $\frac{\vec{a}.\vec{r}}{\vec{a}^2} = \frac{(\vec{a}.\vec{a})\vec{r} - (\vec{a}.\vec{r})\vec{a}}{\vec{a}^2} = \frac{\vec{a}\times(\vec{r}\times\vec{a})}{\vec{a}^2}$

Geometrical Interpretation of $\vec{a}\times(\vec{b}\times\vec{c})$

Consider the expression $\vec{a}\times(\vec{b}\times\vec{c})$ which itself is a vector, since it is a cross product of two vectors $\vec{a}\,\&\,(\vec{b}\times\vec{c})$. Now $\vec{a}\times(\vec{b}\times\vec{c})$ is a vector perpendicular to the plane containing $\vec{a}\,\&\,(\vec{b}\times\vec{c})$ but $\vec{b}\times\vec{c}$ is a vector perpendicular to the plane $\vec{b}\ \&\ \vec{c}$, therefore $\vec{a}\times(\vec{b}\times\vec{c})$ is a vector lies in the plane of $\vec{b}\ \&\ \vec{c}$ and perpendicular to $\vec{a}$. Hence we can express $\vec{a}\times(\vec{b}\times\vec{c})$ in terms of $\vec{b}\ \&\ \vec{c}$ i.e. $\vec{a}\times(\vec{b}\times\vec{c}) = x\vec{b} + y\vec{c}$ where x & y are scalars.

Note :

(i) $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

(ii) $(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}$

(iii) $(\vec{a}\times\vec{b})\times\vec{c} \neq \vec{a}\times(\vec{b}\times\vec{c})$

Ex.29 Find a vector $\vec{v}$ which is coplanar with the vectors $\hat{i}+\hat{j}-2\hat{k}$ & $\hat{i}-2\hat{j}+\hat{k}$ and is orthogonal to the vector $-2\hat{i}+\hat{j}+\hat{k}$. It is given that the projection of $\vec{v}$ along the vector $\hat{i}-\hat{j}+\hat{k}$ is equal to $6\sqrt{3}$.

Sol. A vector coplanar with $\vec{a}\ \&\ \vec{b}$ and orthogonal to $\vec{c}$ is parallel to the triple product,

$(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}$

Hence $\vec{v} = \alpha[(-3(\hat{i}-2\hat{j}+\hat{k}) + 3(\hat{i}+\hat{j}-2\hat{k})] = 9\alpha(\hat{j}-\hat{k})$

Projection of $\vec{v}$ along $\hat{i}-\hat{j}+\hat{k} = \frac{\vec{v}.(\hat{i}-\hat{j}+\hat{k})}{|\hat{i}-\hat{j}+\hat{k}|} = 6\sqrt{3}$

$9\alpha(\hat{j}-\hat{k}).(\hat{i}-\hat{j}+\hat{k}) = 189\alpha(-1,-1) = 18 \Rightarrow \alpha = -1$ Ans. : $9(-\hat{j}+\hat{k})$

Ex.30 ABCD is a tetrahedron with A(–5, 22, 5); B(1, 2, 3); C(4, 3, 2); D(–1, 2, –3). Find $\vec{AB}\times(\vec{BC}\times\vec{BD})$. What can you say about the values of $(\vec{AB}\times\vec{BC})\times\vec{BD}$ and $(\vec{AB}\times\vec{BD})\times\vec{BC}$. Calculate the volume of the tetrahedron ABCD and the vector area of the triangle AEF where the quadrilateral ABDE and quadrilateral ABCF are parallelograms.

Sol. $\vec{AB}\times(\vec{BC}\times\vec{BD}) = 0$; $(\vec{AB}\times\vec{BC})\times\vec{BD} = 0$; $(\vec{AB}\times\vec{BD})\times\vec{BC} = 0$;

Note that $\vec{AB}$; $\vec{BC}$; $\vec{BD}$ are mutually perpendicular.

Volume = $\frac{1}{6}[\vec{AB}, \vec{BC}, \vec{BD}] = \frac{220}{3}$ cu. units

Vector area of triangle AEF $= \frac{1}{2}\vec{AF}\times\vec{AE} = \frac{1}{2}\vec{BC}\times\vec{BD} = -3\hat{i}+10\hat{j}+\hat{k}$

J. PRODUCT OF FOUR VECTORS

(a) SCALAR PRODUCT OF FOUR VECTORS : The products already considered are usually sufficient for practical applications. But we occasionally meet with products of four vectors of the following types. Consider the scalar product of $\vec{a} \times \vec{b}$ and $\vec{c} \times \vec{d}$. This is a number easily expressible in terms of the scalar products of the individual vectors. For, in virtue of the fact that in a scalar triple product the dot and cross may be interchanged, we may write

$(\vec{a} \times \vec{b}).(\vec{c} \times \vec{d}) = \vec{a} \cdot \vec{b} \times (\vec{c} \times \vec{d}) = \vec{a} \cdot ((\vec{b} \cdot \vec{d})\vec{c} - (\vec{b} \cdot \vec{c})\vec{d}) = (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c})$

Writing this result in the form of a determinant, we have $(\vec{a} \times \vec{b}).(\vec{c} \times \vec{d}) = \begin{vmatrix} \vec{a}.\vec{c} & \vec{a}.\vec{d} \\ \vec{b}.\vec{c} & \vec{b}.\vec{d} \end{vmatrix}$

(b) VECTOR PRODUCT OF FOUR VECTORS : Consider next the vector product of $\vec{a} \times \vec{b}$ and $\vec{c} \times \vec{d}$. This is a vector at right angles to $\vec{a} \times \vec{b}$, and therefore coplanar with $\vec{a}$ and $\vec{b}$. Similarly it is coplanar with $\vec{c}$ and $\vec{d}$. It must therefore be parallel to the line of intersection of a plane parallel to $\vec{a}$ and $\vec{b}$ with another parallel to $\vec{c}$ and $\vec{d}$.

To express the product in $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$ in terms of $\vec{a}$ and $\vec{b}$, regard it as the vector triple product of $\vec{a}$, $\vec{b}$ and $\vec{m}$, where $\vec{m} = \vec{c} \times \vec{d}$. Then

$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = (\vec{a} \times \vec{b}) \times \vec{m} = (\vec{a} \cdot \vec{m})\vec{b} - (\vec{b} \cdot \vec{m})\vec{a} = [\vec{a}\,\vec{c}\,\vec{d}]\vec{b} - [\vec{b}\,\vec{c}\,\vec{d}]\vec{a}$ ......(1)

Similarly, regarding it as the vector product of $\vec{n}$, $\vec{c}$ and $\vec{d}$, where $\vec{n} = \vec{a} \times \vec{b}$, we may write it

$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \vec{n} \times (\vec{c} \times \vec{d}) = (\vec{n} \cdot \vec{d})\vec{c} - (\vec{n} \cdot \vec{c})\vec{d} = [\vec{a}\,\vec{b}\,\vec{d}]\vec{c} - [\vec{a}\,\vec{b}\,\vec{c}]\vec{d}$ .....(2)

Equating these two expressions we have a relation between the four vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ viz.

$$[\vec{b}\,\vec{c}\,\vec{d}]\vec{a} - [\vec{a}\,\vec{c}\,\vec{d}]\vec{b} + [\vec{a}\,\vec{b}\,\vec{d}]\vec{c} - [\vec{a}\,\vec{b}\,\vec{c}]\vec{d} = 0 \qquad ......(3)$$

Writing $\vec{r}$ instead of $\vec{d}$, we may express any vector $\vec{r}$ in terms of three other vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ in the form $\vec{r} = \dfrac{[\vec{r}\,\vec{b}\,\vec{c}]\vec{a} + [\vec{r}\,\vec{c}\,\vec{a}]\vec{b} + [\vec{r}\,\vec{a}\,\vec{b}]\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}$, ...(4)

which is valid except when the denominator $[\vec{a}\,\vec{b}\,\vec{c}]$ vanishes, that is except when $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar.

Ex.31 Show that , $(\vec{b} \times \vec{c}) \times (\vec{c} \times \vec{a}) = [\vec{a}\,\vec{b}\,\vec{c}]\,\vec{c}$ and deduce that $[\vec{a} \times \vec{b}\ ,\ \vec{b} \times \vec{c}\ ,\ \vec{c} \times \vec{a}] = [\vec{a}\,\vec{b}\,\vec{c}]^2$.

Sol. L.H.S. : $(\vec{b} \times \vec{c}) \times \vec{u} = (\vec{b}\,.\,\vec{u})\,\vec{c} - (\vec{c}\,.\,\vec{u})\,\vec{b} = [\vec{b}\,\vec{c}\,\vec{a}]\vec{c} - 0$ $\quad (\vec{u} = \vec{c} \times \vec{a})$

Hence $(\vec{b} \times \vec{c}) \times (\vec{c} \times \vec{a}) = [\vec{a}\,\vec{b}\,\vec{c}]\,\vec{c}$ taking dot with $\vec{a} \times \vec{b}$, $[\vec{a} \times \vec{b}\ ,\ \vec{b} \times \vec{c}\ ,\ \vec{c} \times \vec{a}] = [\vec{a}\,\vec{b}\,\vec{c}]^2$.

Ex.32 Show that $\vec{a} \times ((\vec{q} \times \vec{c}) \times (\vec{p} \times \vec{b})) = \vec{b} \times ((\vec{p} \times \vec{c}) \times (\vec{q} \times \vec{a})) + \vec{c} \times ((\vec{p} \times \vec{a}) \times (\vec{q} \times \vec{b}))$

Sol. consider $\vec{a} \times \left[(\vec{q} \times \vec{c}) \times (\vec{p} \times \vec{b})\right] = \vec{a} \times \left[(\vec{u}\,.\,\vec{b})\vec{p} - (\vec{u}\,.\,\vec{p})\vec{b}\right] = (\vec{a} \times \vec{p}).[\vec{q}\,\vec{c}\,\vec{b}] - (\vec{a} \times \vec{b}).[\vec{q}\,\vec{c}\,\vec{b}]$ ......(1)

similarly $\vec{b} \times \left[(\vec{p} \times \vec{c}) \times (\vec{q} \times \vec{a})\right] = (\vec{b} \times \vec{q}).[\vec{p}\,\vec{c}\,\vec{a}] - (\vec{b} \times \vec{a}).[\vec{p}\,\vec{c}\,\vec{q}]$ ....(2)

and $\vec{c} \times \left[(\vec{p} \times \vec{a}) \times (\vec{q} \times \vec{b})\right] = \vec{c} \times \left[\vec{u} \times \vec{v}\right] = (\vec{c}.\vec{v})\,\vec{u} - (\vec{c}\,.\,\vec{u})\,\vec{v} = [\vec{c}\,\vec{q}\,\vec{b}]\,(\vec{p} \times \vec{a}) - [\vec{c}\,\vec{p}\,\vec{a}] \times (\vec{q} \times \vec{b})$ ....(3)

Now (1) – (2) – (3) = 0 $\Rightarrow$ result

K. VECTOR EQUATIONS

Ex.33 Solve the equation $\vec{x} \times \vec{a} = \vec{b}, (\vec{a} \cdot \vec{b} = 0)$.

Sol. From the vector product of each member with a, and obtain $\vec{a}^2\vec{x} - (\vec{a} \cdot \vec{x})\,\vec{a} = \vec{a} \times \vec{b}$.

The general solution, with λ as parameter, is $\vec{x} = \lambda\vec{a} + \vec{a} \times \vec{b}/\vec{a}^2$.

Ex.34 Solve the simultaneous equations $p\vec{x} + q\vec{y} = \vec{a}, \vec{x} \times \vec{y} = \vec{b}, (\vec{a} \cdot \vec{b} = 0)$.

Sol. Multiply the first vectorially by $\vec{x}$, and substitute for $\vec{x} \times \vec{y}$ from the second. Then $q\vec{b} = \vec{x} \times \vec{a}$, which is of the same form as the equation in the preceding example. Thus $\vec{x} = \lambda\vec{a} + q\vec{a} \times \vec{b}/\vec{a}^2$.

Substitution of this value in the first equation gives $\vec{y}$.

Ex.35 Find $\vec{x}$ so as to satisfy both the equations $\vec{x} \times \vec{a} = \vec{b}, \quad \vec{x} \cdot \vec{c} = p \qquad (\vec{a} \cdot \vec{b} = 0)$ .....(i)

Sol. Multiply the first vectorially by $\vec{c}$, expand, and use the second. Then $(\vec{c} \cdot \vec{a})\vec{x} - p\vec{a} = \vec{c} \times \vec{b}$.

Thus $\vec{x} = (p\vec{a} + \vec{c} \times \vec{b})/\vec{a} \cdot \vec{c}$ ....(ii)

provided $\vec{a} \cdot \vec{c} \neq 0$. If, however, $\vec{a} \cdot \vec{c} = 0$, use the general solution of the first equation (i).

$x = \lambda\vec{a} + \vec{a} \times \vec{b}/\vec{a}^2$ ....(iii)

This will satisfy the second equating (i) for any value of λ provided $pa^2 = \vec{a} \times \vec{b} \cdot \vec{c}$, which is a necessary condition when $\vec{a} \cdot \vec{c} = 0$.

Ex.36 Solve $\vec{x} + \vec{a} + (\vec{x} \cdot \vec{b})\vec{c} = \vec{d}$ ....(i)

Sol. Multiply scalarly by $\vec{a}$. Then $(\vec{x} \cdot \vec{b})(\vec{a} \cdot \vec{c}) = \vec{a} \cdot \vec{d}$. Substitute for $\vec{x} \cdot \vec{b}$ in (i) and obtain

$$\vec{x} \times \vec{a} = \vec{d} - (\vec{a} \cdot \vec{d})\vec{c}/\vec{a} \cdot \vec{c} = \vec{a} \times (\vec{d} \times \vec{c})/\vec{a} \cdot \vec{c}.$$

$$\vec{x} = \lambda\vec{a} + \vec{a} \times (\vec{a} \times (\vec{d} \times \vec{c}))\,/\,(\vec{a} \cdot \vec{c})\vec{a}^2.$$

Ex.37 Solve $p\vec{x} + \vec{x} \times \vec{a} = \vec{b}, (p \neq 0)$ ....(i)

Sol. Multiply scalarly by a. Then $p\vec{x} \cdot \vec{a} = \vec{b} \cdot \vec{a}$ ....(ii)

Multiply (i) vectorially by $\vec{a}$, expand the triple product, and substitute for $\vec{x} \cdot \vec{a}$ from (ii). Then

$$p^2\vec{x} \times \vec{a} + (\vec{b} \cdot \vec{a})\vec{a} - p\vec{a}^2\vec{x} = p\vec{b} \times \vec{a}.$$

Eliminate $\vec{x} \times \vec{a}$ between this equation and (i), and find $\vec{x} = (p^2\vec{b} + (\vec{b} \cdot \vec{a})\vec{a} - p\vec{a}^2\,\vec{x} = p\vec{b} \times \vec{a}$.

Ex.38 If $\vec{A} + \vec{B} = \vec{a}, \vec{A} . \vec{a} = 1$ and $\vec{A} \times \vec{B} = \vec{b}$ then prove that $\vec{A} = \dfrac{\vec{a} \times \vec{b} + \vec{a}}{|\vec{a}|^2}$ and $\vec{B} = \dfrac{\vec{b} \times \vec{a} + \vec{a}\left(|\vec{a}|^2 - 1\right)}{|\vec{a}|^2}$

Sol. $\vec{A} + \vec{B} = \vec{a}$ taking dot with $\vec{a}$ $\quad \vec{a}.\vec{B} = |\vec{a}|^2 - 1$ ....(1)

$\vec{A} \times \vec{B} = \vec{b}$ taking cross with $\vec{a}$ $\quad \left(\vec{a}.\vec{B}\right)\vec{A} - \left(\vec{a}.\vec{A}\right)\vec{B} = \vec{a} \times \vec{b} \quad \left(|\vec{a}|^2 - 1\right)\vec{A} - \vec{B} = \vec{a} \times \vec{b}$ ....(2)

Solving (2) and $\vec{A} + \vec{B} = \vec{a}$ simultaneously we get the desired result.

Ex.39 Solve the vector equation in $\vec{x}$: $\quad \vec{x} + \vec{x} \times \vec{a} = \vec{b}$.

Sol. Taking dot with $\vec{a}$ $\quad \vec{x}.\vec{a} = \vec{b}.\vec{a}$ ....(1)

Taking cross with $\vec{a}$ $\quad \vec{x} \times \vec{a} + (\vec{x} \times \vec{a}) \times \vec{a} = \vec{b} \times \vec{a}$ ....(2)

$$\vec{b} - \vec{x} + (\vec{x}.\vec{a})\vec{a} - (\vec{a}.\vec{a})\vec{z} = \vec{b} \times \vec{a} \quad (\vec{b} + \vec{a}(\vec{b}.\vec{a}) - \vec{b} \times \vec{a} = \vec{x}(1 + \vec{a}.\vec{a}) \quad \Rightarrow \quad \vec{x} = \frac{1}{1 + \vec{a}.\vec{a}}\left\{\vec{b} + (\vec{a}.\vec{b})\vec{a} + \vec{a} \times \vec{b}\right\}$$

Ex.40 Express a vector $\vec{R}$ as a linear combination of a vector $\vec{A}$ and another perpendicular to $\vec{A}$ and coplanar with $\vec{R}$ and $\vec{A}$.

Sol. $\vec{A} \times (\vec{A} \times \vec{R})$ is a vector perpendicular to $\vec{A}$ and coplanar with $\vec{A}$ and $\vec{R}$.

Hence let, $\vec{R} = \lambda\vec{A} + \mu\vec{A} \times (\vec{A} \times \vec{R})$ ....(1)

taking dot with $\vec{A}$, $\vec{R}.\vec{A} = \lambda\vec{A}.\vec{A}$ $\Rightarrow$ $\lambda = \dfrac{\vec{R}.\vec{A}}{\vec{A}.\vec{A}}$

again taking cross with $\vec{A}$ $\vec{R} \times \vec{A} = \mu[\vec{A} \times (\vec{A} \times \vec{R})] \times \vec{A}$ $= \mu[(\vec{A}.\vec{R})\vec{A} - (\vec{A}.\vec{R})\vec{R}] \times \vec{A}$

$= -\mu(\vec{A}.\vec{A})(\vec{R}.\vec{A})$ $\therefore$ $\mu = -\dfrac{1}{\vec{A}.\vec{A}}$ Hence $\vec{R} = \left(\dfrac{\vec{R}.\vec{A}}{\vec{A}.\vec{A}}\right)\vec{A} - \dfrac{1}{\vec{A}.\vec{A}}\ \vec{A} \times (\vec{A} \times \vec{R})]$

L. EQUATION OF A PLANE

Consider the plane through a given point $\vec{a}$, and perpendicular to the vector $\vec{m}$. If $\vec{r}$ is the position vector of a point P on it, $\vec{r} - \vec{a}$ is parallel to the plane and therefore perpendicular to $\vec{m}$. Consequently $(\vec{r} - \vec{a}).\vec{m} = 0$ ....(1)

This equation is satisfied by any point on the plane, but by no point off the plane.

Let $\hat{n}$ denote the unit vector perpendicular to this plane, and directed from the origin toward the plane. Then the above equations equivalent to $(\vec{r} - \vec{a}).\hat{n} = 0$. ....(2)

The scalar product $\vec{a}.\hat{n}$ is the resolute of $\overrightarrow{OA}$ along $\hat{n}$, and is equal to the perpendicular ON from the origin to the plane. This is positive, and will be denoted by p. We may then write the equation of the plane in the form $p - \vec{r}.\hat{n} = 0$ ....(3)

which will be called the normal form of the equation of the plane. It should be remembered that, in this form, $\hat{n}$ has the sense from the origin to the plane, and p is positive.

The inclination of two planes, whose equations are $p - (\vec{r}.\hat{n}) = 0, p' - (\vec{r}.\hat{n}') = 0$ is the angle θ between their normals; and this is given by $\cos\theta = \hat{n}.\hat{n}'$.

Distance of a point from a plane : It is required to find the perpendicular distance from a point P'(r') to the plane (3), measured in the sense of $\hat{n}$ above, Consider the parallel plane through P' and let p' be the perpendicular from O to this plane. Then, as above, $p' = \vec{r}'.\hat{n}$, and the perpendicular distance from P' to the given plane is $p - p' = p - \vec{r}'.\hat{n}$ ....(1)

This is positive for points on the same side of the plane as the origin, negative for points on the opposite sides.

To find the distance from P' to the plane, measured i the direction of the unit vector $\hat{b}$, let $\vec{a}$ parallel to $\hat{b}$ drawn through P' cut the plane in H. Then if d is the length of P'H, the position vector of H is $\vec{r}' + d\vec{b}$; and, since this point lies on the given plane, we have $p - (\vec{r}' + d\vec{b}).\hat{n} = 0$,

so that $d = (p - \vec{r}'.\hat{n}) / \vec{b}.\hat{n}$.

We find the equations of the planes which bisect the angles between two given planes

$p - \vec{r}.\hat{n} = 0.$ $p' - \vec{r}.\hat{n}' = 0.$

These follows from the fact that a point on either bisector is equidistant from the two planes. For points on the plane bisecting the angle in which the origin lies, the two perpendicular distance have the same signs. but for points $\hat{n}$ the other bisector, opposite signs. The equation of the former bisector is therefore $p - \vec{r}.\hat{n} = p' - \vec{r}.\hat{n}'$, or $p - p' = \vec{r}.(\hat{n} - \hat{n}')$

and that of the other bisector is $p + p' = \vec{r}.(\hat{n} + \hat{n}')$

Angle between the 2 planes is the angle between 2 normals drawn to the planes and the angle between a line and a plane is the compliment of the angle between the line and the normal to the plane.

Ex.41 If any point O within or without a tetrahedron ABCD is joined to the vertices, and AO, BO, CO, DO are produced to cut the planes of the opposite faces in P, Q, R, S respectively, then $\Sigma \frac{OP}{AP} = 1$.

Sol. With O as origin let the position vectors of A, B, C, D be $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ respectively. Any one of these vectors may be expressed in terms of the other three, so that there is a linear relation connecting them which may be written $\ell\vec{a} + m\vec{b} + n\vec{c} + p\vec{d} = 0$. ....(1)

The equation of the line AP is $\vec{r} = -u\vec{a}$, u being positive for points of the line which lie on the oposite side of the origin from A. In virtue of (1) we may write this equation as

$$\ell\vec{r} - u(m\vec{b} + n\vec{c} + p\vec{d}) = 0 \qquad ....(2)$$

For the particular point P, in which this line cuts the plane BCD, the four points $\vec{r}, \vec{b}, \vec{c}, \vec{d}$ are coplanar. Hence the sum of the coefficients in (2) is zero, so that $u = \ell/(m + n + p)$, and therefore

$$\frac{OP}{AP} = \frac{u}{1+u} = \frac{\ell}{\ell+m+n+p}.$$

the other ratio may be written down by cyclic permutation of the symbols, and their sum is obviously equal to unity.

Ex.42 Find the intersection of the line joining the points (1, –2, –1) and (2, 3, 1) with the plane through the points (2, 1, –3), (4, –1, 2) and (3, 0, 1).

Sol. The equation of the straight line is (x, y, z) = (1, –2, –1) + t(1, 5, 2) and that of the plane is (x, y, z) = (2, 1, –3) + u(2, –2, 5) + v(1, –1, 4).
For the point of intersection the two equation given the same values of x, y, z. Hence on equating corresponding components in the two expressions for (x, y, z) we find

$$1 + t = 2 + 2u + v, \quad -2 + 5t = 1 - 2u - v,$$

and a third equation. From the first two by addition we find t = 2/3, so that the point of intersection is (5, 4, 1)/3.

PLANES SATISFYING VARIOUS CONDITIONS : We shall now see how the triple products may be used in finding the equation of a plane subject to certain conditions. Let us examine the following typical cases

(i) Plane through three given points A, B, C : Let $\vec{a}, \vec{b}, \vec{c}$ be the position vectors of the points relative to an assigned origin O, and $\vec{r}$ that of a variable point P on the plane. Since P, A, B, C all lie on the plane, the vectors $\vec{r} - \vec{a}$, $\vec{a} - \vec{b}$, $\vec{b} - \vec{c}$ are coplanar, and their scalar triple product is zero.
Hence $(\vec{r} - \vec{a}) \cdot ((\vec{a} - \vec{b}) \times (\vec{b} - \vec{c})) = 0$
If two expand this, and neglect the triple products in which any vector occurs twice, the equation becomes $\vec{r} \cdot (\vec{b} \times \vec{c} + \vec{c} \times \vec{a} + \vec{a} \times \vec{b}) = [\vec{a}\,\vec{b}\,\vec{c}]$.
Thus the plane is perpendicular to the vector area of the triangle ABC.

(ii) Plane through a given point parallel to two given straight lines : Let $\vec{a}$ be the given point, and $\vec{b}, \vec{c}$ two vectors parallel to the given lines. Then $\vec{b} \times \vec{c}$ is perpendicular to the plane; and we have only to write down the equation of the plane through a perpendicular to $\vec{b} \times \vec{c}$ is
$(\vec{r} - \vec{a}) \cdot \vec{b} \times \vec{c} = 0$, i.e. $\vec{r} \cdot \vec{b} \times \vec{c} = [\vec{a}\,\vec{b}\,\vec{c}]$.

(iii) Plane containing a given straight line and parallel to another : Let the first line be represented by $\vec{r} = \vec{a} + t\vec{b}$, while the second is parallel to $\vec{c}$. Then the plane in question contains the point $\vec{a}$, and is parallel to $\vec{b}$ and $\vec{c}$. its equation is therefore, by the last case $\vec{r} \cdot (\vec{b} \times \vec{c}) = [\vec{a}\,\vec{b}\,\vec{c}]$.

(iv) Plane through two given points and parallel to a given straight line : Let $\vec{a}, \vec{b}$ be two given points, and $\vec{c}$ a vector parallel to the given straight line. The required plane then passes through $\vec{a}$ and is parallel to $\vec{b} - \vec{a}$ and $\vec{c}$. Its equation is therefore $\vec{r} \cdot (\vec{b} - \vec{a}) \times \vec{c} = [\vec{a}, \vec{b} - \vec{a}, \vec{c}] = [\vec{a}\,\vec{b}\,\vec{c}]$.

(v) Plane containing a given straight line and a given point : Let $\vec{r} = \vec{a} + t\vec{b}$ be the given straight line and $\vec{c}$ the given point. Then the plane in question passes through the two points $\vec{a}, \vec{c}$ and is parallel to $\vec{b}$. Hence by (iv) is its equation is $\vec{r} \cdot (\vec{a} - \vec{c}) \times \vec{b} = [\vec{a}\,\vec{b}\,\vec{c}]$.

CONDITION OF INTERSECTION OF TWO STRAIGHT LINES :

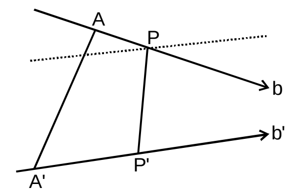
Let the equations of the given straight lines be $\vec{r} = \vec{a} + t\vec{b}$, $r = \vec{a}' + s\vec{b}'$, so that they pass through the points $\vec{a}$, $\vec{a}'$ and are parallel to $\vec{b}$, $\vec{b}'$ respectively. If they intersect, their common plane must be parallel to each of the vectors $\vec{b}$, $\vec{b}'$, $\vec{a} - \vec{a}'$, whose scalar triple product is therefore zero. Hence the required condition is $[\vec{b}, \vec{b}', \vec{a} - \vec{a}'] = 0$

THE COMMON PERPENDICULAR TO TWO SKEW LINES :

Let the equation of the two straight lines be $\vec{r} = \vec{a} + t\vec{b}$; $\vec{r} = \vec{a}' + s\vec{b}'$

Then the vector $\vec{n} = \vec{b} \times \vec{b}'$ is perpendicular to both lines, and therefore parallel to their common perpendicular P'P. If A, A' are the points $\vec{a}$, $\vec{a}'$ respectively, the length p of this common perpendiuclar is equal to the length of the projection of A'A on $\vec{n}$. Hence $p = \dfrac{\vec{n}.(\vec{a}-\vec{a}')}{n} = \dfrac{1}{n}[\vec{b}, \vec{b}', \vec{a} - \vec{a}']$

The equation of the plane containing the first line and the common perpendicular to the two lines is $[\vec{r} - \vec{a}', \vec{b}', \vec{b} \times \vec{b}'] = 0$. The point P' is that in which the second line meets this plane. Similarly the equation of the plane containing the second line and the common perpendicular is $[\vec{r} - \vec{a}', \vec{b}', \vec{b} \times \vec{b}'] = 0$. These two planes determine the line P'P.



Ex.43 Find the shortest distance between the straight lines through the points A(6, 2, 2) and A'(–4, 0, –1) in the direction (1, –2, 2) and (3, –2, –2) respectively. Also find the feet, P and P', of the common perpendicular.

Sol. In this case $\vec{b} \times \vec{b}' = 4(2, 2, 1)$, and the unit vector in this direction is (2, 2, 1)/3. The shortest distance is the projection of A'A on this direction, so that p = (10, 2, 3). (2, 2, 1)/3 = 27/3 = 9.
The pair of skew lines is therefore right-handed. The equation of the plane APP' is 2x – y – 2z = 6; and the second line meets this in the point P' (–1, –2, –3). Shown similarly that P is the point (5, 4, 0).
Also show that the moment about either line, of a unit vector localized in the other, is $36/\sqrt{17}$.

Ex.44 Examine similarly the pair of line determined by the equations
3x – 4y – z + 5 = 0 = 3x – 6y – 2z + 13 and 3x + 4y – 3z + 2 = 0 = 3x – 2y + 6z + 17.

Sol. The first line, being the intersection of two planes, is perpendicular to both normals, and therefore has the direction of the vector $\vec{b} = (2, 3, -6)$. One point on the line is A(5, 6, –4). Similarly the second line has the direction of $\vec{b}' = (2, -3, -2)$, and one point on the line is A'(1, –5, –5). These directions along the lines are chosen because they are inclined at an acute angle. Then $\vec{b} \times \vec{b}' = -4(6, 2, 3)$, and the unit vector in this direction is –(6, 2, 3)/7. The projection of A'A on this direction is then

p = –(6, 2, 3). (4, 11, 1)/7 = –49/7 = –7.

The pair of lines is therefore left-handed. The equation of the plane containing the second lines and the common perpendicular is 5x + 18y – 22z = 25;
and this plane is cut by the first line in P(3, 3, 2). Similarly find P'(–3, 1, –1). Also show that the moment about either line, of a unit vector localized in the other, is $-28/\sqrt{17}$.

Ex.45 Find the equation of the plane through the line $\vec{d}$, $\vec{m}$ parallel to $\vec{c}$.

Sol. Since the required plane is parallel to $\vec{d}$ and $\vec{c}$, its normal is parallel to $\vec{d} \times \vec{c}$. If $\vec{b}$ is a point on the given line, then $\vec{b} \times \vec{d} = \vec{m}$, and, $\vec{r}$ being a current point on the plane, $\vec{r} - \vec{b}$ is parallel to the plane, and therefore perpendicular to the normal. Hence $(\vec{r} - \vec{b}).(\vec{d} \times \vec{c}) = 0$,
which may be written $[\vec{r}\ \vec{d}\ \vec{c}] = \vec{m} \cdot \vec{c}$. This is the required equation of the plane.

Ex.46 Find the equation of the plane through the point $\vec{a}$ and the line $\vec{d}$, $\vec{m}$.

Sol. If $\vec{b}$ is a point on the given line, then $\vec{b} \times \vec{d} = \vec{m}$. The plane is parallel to $\vec{b} - \vec{a}$ and $\vec{d}$, and its normal is parallel to $(\vec{b} - \vec{a}) \times \vec{d}$. Hence if $\vec{r}$ is a current point on the plane $(\vec{r} - \vec{a})$ is also parallel to the plane, and therefore $(\vec{r} - \vec{a}).(\vec{b} - \vec{a}) \times \vec{d} = 0$, which may be written $\vec{r} \cdot \vec{m} - [\vec{r}\ \vec{a}\ \vec{d}] = \vec{a} \cdot \vec{m}$.